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LETTER TO THE EDITOR

The temperature effect in the Casimir attraction of a thin metal film

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Abstract. The Casimir effect for conductors at arbitrary temperatures is studied theoretically. By using the analytical properties of the Green functions and applying the Abel–Plana formula to Lifshitz's equation, the Casimir force is presented as a sum of temperature dependent and vacuum contributions of the fluctuating electromagnetic field. The general results are applied to the system consisting of a bulk conductor and a thin metal film. It is shown that the characteristic frequency of the thermal fluctuations in this system is proportional to the square root of the thickness of the metal film. For the case of sufficiently high temperatures when the thermal fluctuations play the main role in the Casimir interaction, this leads to the growth of the effective dielectric permittivity of the film and to the disappearance of the dependence of Casimir's force on the sample thickness.

1. Introduction

The theoretical and experimental study of the Casimir effect has more than fifty years of history (see, for example, reviews [1, 2]). The Casimir effect for plane-parallel dielectric surfaces was worked out by Lifshitz [3] and Schwinger *et al* [4, 5]. The kernel of this phenomenon consists in the fluctuation electromagnetic interaction of uncharged bodies. The Casimir energy of a dielectric ball in vacuum was evaluated in Refs. [6–9]. Different methods have been used for dealing with the Casimir effect: the complex contour integration method [7, 10], the Hurwitz zeta function method [10, 11], and the zeta function technique in combination with the contour integral representations [12, 13].

Measurements of the Casimir force between metal bodies was performed with a good experimental accuracy only recently [14, 15]. For metals with a high value of the conductivity σ, the Casimir interaction manifests itself as the attractive force f_0 (per unit area) which varies as the inverse fourth power of the distance a between the plates [16]:

$$f_0 = -\frac{\pi^2}{240}\frac{\hbar c}{a^4} \qquad (T = 0, \sigma \to \infty) \tag{1}$$

where c is the speed of light and T is the temperature.

With an increase of the temperature, but for

$$\frac{kT}{\hbar} \ll \frac{c}{a} \tag{2}$$

an additional term proportional to the fourth power of the temperature appears in the Casimir force (see [2] and references therein):

$$\Delta f(T) = -\frac{\pi^2}{45}\frac{(kT)^4}{(\hbar c)^3}. \tag{3}$$

This term was derived under an assumption that thermal equilibrium between the matter and the radiation occurs. The additional a-independent attractive force between the metal plates arises as a result of the pressure of the thermal radiation outside the plates. Under the inverse inequality

$$\frac{kT}{\hbar} \gg \frac{c}{a} \tag{4}$$

the Casimir force is completely defined by the temperature and is described by the formula

$$f(T) = -\zeta(3)\frac{kT}{8\pi a^3} \tag{5}$$

with exponential accuracy [3, 17–19]. Here $\zeta(x)$ is the zeta function, $\zeta(3) \approx 1.202$. For $T = 300$ K, the parameter $\hbar c/kT$ is about 10 μm. So, the Casimir force between the bulk metal plates displays a weak temperature dependence (equation (3)) in the range of realistic separations a of about 0.1–1 μm.

The temperature effects in the Casimir force could be brought to the forefront if the interacting objects are thin metal films. Indeed, formula (1) is obtained under an assumption that the thickness d of the plates is the greatest parameter with the dimension of length. As was shown in [20, 21], the asymptotical formula (1) appears to be invalid for thin metal plates if the inequality

$$\omega_p\sqrt{\frac{d}{a}} = \omega_c \ll \omega_p, \frac{c}{a} \tag{6}$$

is satisfied. Here ω_p is the plasma frequency and ω_c is the characteristic frequency of the fluctuating electromagnetic field. In this case the collective properties of the electron subsystem of the metal are important in forming the Casimir force. Specifically, the evaluation

$$f_0 \propto -\frac{\hbar\omega_c^2}{\nu + \omega_c}\frac{1}{a^3} \qquad (T = 0) \tag{7}$$

is valid if the metal film of thickness d with a weak reflecting power interacts with the bulk metal (ν is the frequency of the electron bulk collisions). A decrease of the absolute value of the Casimir force $f(T = 0)$ makes the temperature dependence $f(T)$ pronounced even in the range of realistic separations $a \sim 0.1$–1 μm. The study of this dependence is the subject of this letter.

2. Statement of the problem. The basic equations

The general formula for the force of Casimir interaction between dielectric slabs with arbitrary dielectric constants ϵ was originally derived by Lifshitz [3] (see, also, [4, 5, 22, 23]). The Casimir force is presented in this formula as a functional defined on the set of functions $\epsilon(\mathrm{i}\omega_n)$ of a discrete variable $\omega_n = 2\pi nkT$ ($n = 0, 1, 2, \ldots$). We use Lifshitz's formula for the system comprising a bulk conductor and a thin metal film of thickness d separated by a distance a. The system of coordinates is chosen such that the x-axis is perpendicular to the plane of interacting plates. The conductivity of our system as a function of the coordinate x is

$$\sigma(x) = \sigma[\theta(-x + a + d) - \theta(-x + a)] + \sigma_\infty\theta(x) = \begin{cases} 0 & x \in S_{\mathrm{I}} \qquad x \in S_{\mathrm{III}} \\ \sigma & x \in S_{\mathrm{II}} \\ \sigma_\infty & x \in S_{\mathrm{IV}} \end{cases} \tag{8}$$

where $\theta(x)$ is the Heaviside function, $S_{\mathrm{I}} = (-\infty, -a-d)$ and $S_{\mathrm{III}} = (-a, 0)$ are vacuum domains and $S_{\mathrm{II}} = (-a-d, -a)$ and $S_{\mathrm{IV}} = (0, \infty)$ are regions occupied by the metal film and the bulk conductor, respectively. We do not make any specific supposition about the conductivity σ_∞ of the bulk conductor because its value does not affect the final result. As for the conductivity of the metal film, we take it in the τ-approximation with the following frequency dependence:

$$\sigma(\omega) = \frac{\omega_p^2}{\nu - \mathrm{i}\omega}. \tag{9}$$

Lifshitz's formula for the Casimir force can be expressed in terms of the conductivity (8) of our system:

$$F[\sigma] = -kT \sum_{n=0}^{\infty}{}' \int \mathrm{d}\vec{r} \frac{\delta\sigma^{(M)}(x|\omega_n)}{\delta a} \Gamma_{ii}^{(M)}(\vec{r}, \vec{r}|\omega_n) \tag{10}$$

where Matsubara's conductivity $\sigma^{(M)}(\omega_n)$ is related to the frequency dispersion of the metal conductivity $\sigma(\omega)$ by the espression

$$\sigma^{(M)}(\omega_n) = \sigma(\mathrm{i}\omega_n). \tag{11}$$

$\Gamma^{(M)}$ is the temperature Green function of the electromagnetic field; the prime on the sum symbol indicates that the term with $n = 0$ is taken with half the weight. By using the analytical properties of the Green functions and the Abel–Plana formula for summing the series, equation (10) can be rewritten in the integral form (see the appendix):

$$F[\sigma] = -\frac{1}{2\pi} \int \mathrm{d}\vec{r} \left\{ \int_0^\infty \mathrm{d}\zeta \frac{\delta\sigma(x|\mathrm{i}\zeta)}{\delta a} \Gamma_{ii}(\vec{r}, \vec{r}|\mathrm{i}\zeta) \right. \\ \left. +2 \int_0^\infty \mathrm{d}\omega \mathrm{Im} \left[\frac{\delta\sigma(x|\omega)}{\delta a} \Gamma_{ii}(\vec{r}, \vec{r}|\omega) \right] (\mathrm{e}^{\hbar\omega/kT} - 1)^{-1} \right\}. \tag{12}$$

The first term in equation (12) describes the Casimir force at zeroth temperature and is obtained from equation (10) through the simple change of the summation by integrating over the imaginary frequency. The second term provides the temperature-dependent contribution to the Casimir force which is suppressed by the small exponential factor $\exp(-\hbar\omega/kT)$ at $T \to 0$. In contrast to the low-temperature case, this term can be governing in the force (12) at sufficiently high temperatures.

In order to simplify the general formula (10), we introduce the transverse spatial Fourier transformation

$$\Gamma_{ik}^{(M)}(\vec{r}, \vec{r}'|\omega_n) = \int \frac{\mathrm{d}\vec{q}}{(2\pi)^2} \exp[\mathrm{i}\vec{q}(\vec{r} - \vec{r}')_\perp] \Gamma_{ik}^{(M)}(x, x'|q^2, \omega_n).$$

At infinitesimal displacement δa of the bulk conductor, a change

$$\delta\sigma(x) = -\sigma_\infty \delta(x) \delta a$$

of the conductivity $\sigma(x)$ occurs. Therefore, formula (10) for the Casimir force can be rewritten in the final form:

$$f = \frac{F}{A} = kT \int_0^\infty \frac{\mathrm{d}q^2}{4\pi} \sum_{n=0}^{\infty}{}' \sigma_\infty \Gamma_{ii}^{(M)}(x, x'|q^2, \omega_n) \Big|_{x \to 0, x' \to 0} \tag{13}$$

where A is the area of the slabs. We interpret the limiting process in equation (13) as one in which x and x' tend to the interface from opposite sides, $x \to -0$ and $x' \to +0$. The formula (13) defines the force acting on unit area of the bulk conductor from the metal film. The positive force corresponds to the repulsion of bodies and the negative one to the attraction.

In terms of 'transverse electric' and 'transverse magnetic' Green functions g^{e} and g^{m} [4], we have

$$\lim_{x,x'\to 0}\Gamma_{ii}^{(M)}(x,x') = \lim_{x,x'\to 0}\left[\omega_n g^{\mathrm{e}}(x,x') - \omega_n^{-1}\left(\partial_x \frac{1}{\epsilon_\infty}\partial'_x + \frac{q^2}{\epsilon_\infty}\right) g^{\mathrm{m}}(x,x')\right]$$

where g^{e} and g^{m} are defined by

$$[-\partial_x^2 + q^2 + \omega_n^2 \epsilon(x|\omega_n)]g^{\mathrm{e}}(x,x') = \delta(x-x') \tag{14}$$

and

$$\left[-\partial_x \frac{1}{\epsilon(x|\omega_n)}\partial_x + \frac{q^2}{\epsilon(x|\omega_n)} + \omega_n^2\right] g^{\mathrm{m}}(x,x') = \delta(x-x'). \tag{15}$$

Here

$$\epsilon(x|\omega_n) = 1 + \frac{\sigma(x|\mathrm{i}\omega_n)}{\omega_n} \tag{16}$$

is the effective dielectric permittivity of a metal taken at the imaginary frequency.

Thus, to analyse the temperature dependence of the Casimir force we should solve the set of equations (14), (15) and substitute the obtained function $\Gamma_{ii}^{(M)}(x=0, x'=0|q^2,\omega_n)$ into equation (13).

3. The temperature dependence of the Casimir force

While solving the set of equations (14) and (15), we are interested in Green functions with the argument x' within the region S_{IV} occupied by the bulk conductor. For $x' \in S_{\mathrm{IV}}$, the general solutions of equations (14) and (15) have the following form:

$$g_{\mathrm{I}}^{(\mathrm{e,m})} = A\mathrm{e}^{kx-\kappa_\infty x'} \qquad g_{\mathrm{II}}^{(\mathrm{e,m})} = (B_1\mathrm{e}^{\kappa x} + B_2\mathrm{e}^{-\kappa x})\mathrm{e}^{-\kappa_\infty x'}$$
$$g_{\mathrm{III}}^{(\mathrm{e,m})} = (C_1\mathrm{e}^{kx} + C_2\mathrm{e}^{-kx})\mathrm{e}^{-\kappa_\infty x'}$$
$$g_{\mathrm{IV}}^{\mathrm{e}} = (2\kappa_\infty)^{-1}(\mathrm{e}^{-\kappa_\infty|x-x'|} + r\mathrm{e}^{-\kappa_\infty(x+x')})$$
$$g_{\mathrm{IV}}^{\mathrm{m}} = \epsilon_\infty(2\kappa_\infty)^{-1}(\mathrm{e}^{-\kappa_\infty|x-x'|} + r\mathrm{e}^{-\kappa_\infty(x+x')})$$

where

$$k = \sqrt{q^2+\omega_n^2} \qquad \kappa_\infty = \sqrt{q^2+\epsilon_\infty\omega_n^2} \qquad \kappa = \sqrt{q^2+\epsilon\omega_n^2}.$$

Determining constants A, $B_{1,2}$, $C_{1,2}$ and r from the boundary conditions to equations (14) and (15), we obtain for the difference

$$\Gamma_{ii}^{(M)}(a) - \Gamma_{ii}^{(M)}(a\to\infty) \equiv \mathrm{reg}\Gamma_{ii}^{(M)}(a) \tag{17}$$

the expression

$$\mathrm{reg}\Gamma_{ii}^{(M)}(a) = -k\sigma_\infty^{-1}\frac{2\epsilon kd\exp(-2ka)}{2+\epsilon kd[1-\exp(-2ka)]} + \cdots. \tag{18}$$

This formula is derived under inequalities (6) and

$$d \ll \delta_0 = c/\omega_p. \tag{19}$$

Dots in equation (18) denote terms of higher order of the small parameter d/δ_0. The procedure (17) of the regularization of the Green function allows us to avoid the 'surface' divergence in the Casimir force. The divergent term does not depend on the separation a

between the plates and represents an addition to the renormalized Casimir force. According to equations (13) and (18), the expression for this force is

$$f = -kT \int_0^\infty \frac{\mathrm{d}q^2}{2\pi} \sum_{n=0}^{\infty}{}' \frac{\epsilon k^2 d \exp(-2ka)}{2 + \epsilon kd[1 - \exp(-2ka)]}. \tag{20}$$

For Casimir's interaction of sufficiently thin films, the characteristic frequencies of the fluctuating electromagnetic field turn out to be much less than the parameter c/a. In this case, one can neglect the relativistic retarding effect and passage to the limit $c \to \infty$ [24,25]. This allows us to assume $k = q$ in equation (20) for the characteristic frequencies and to approximate the Casimir force as

$$f = -\frac{B}{4\pi\beta a^3} \int_0^\infty \mathrm{d}x\, x^3 I(x) \mathrm{e}^{-x} \qquad \beta = \frac{1}{kT} \tag{21}$$

where $x = 2qa$ is the new variable of integration, symbol $I(x)$ denotes the sum of the series:

$$I(x) = \sum_{n=0}^{\infty}{}' \frac{1}{n(n+C) + BF(x)} \tag{22}$$

with the parameters

$$B = \frac{\omega_p^2 \beta^2}{(4\pi)^2} \frac{d}{a} \qquad C = \frac{\beta\nu}{2\pi}$$

and the function $F(x) = x(1 - \mathrm{e}^{-x})$.

In the case (6), the temperature-dependent part of the Casimir force can be calculated by means of equation (21) without using the summation formula (12) for the Casimir force. However, the analysis of the spectral integrals in (12) is useful in determining the characteristic frequencies which provide the main contribution to the Casimir force f. Let us recall that the contribution f_0 of the vacuum fluctuating electromagnetic field to the Casimir force is defined by the spectral density of energy taken at the imaginary frequency whereas the contribution $\Delta f(T)$ of the thermal radiation of the system is related to the imaginary part of the spectral density at the real frequencies. The force f_0 related to the vacuum fluctuations is evaluated by equation (5). It disappears as the film thickness decreases, $d \to 0$.

Now consider the temperature-dependent part of the Casimir force. According to equations (12) and (21), we have

$$\Delta f(T) = -\frac{BC}{4\pi a^3} \int_0^\infty \mathrm{d}x\, x^3 \mathrm{e}^{-x} \int_0^\infty \mathrm{d}\tau (\mathrm{e}^{2\pi\tau} - 1)^{-1} \frac{\tau}{(\tau^2 - BF(x))^2 + C^2\tau^2} \tag{23}$$

where $\tau = 2\pi\beta\omega$. At $C \to 0$, this expression can be approximated as

$$\begin{aligned} \Delta f(T) &= -\frac{B}{4a^3} \int_0^\infty \mathrm{d}x\, x^3 \mathrm{e}^{-x} \int_0^\infty \mathrm{d}\tau (\mathrm{e}^{2\pi\tau} - 1)^{-1} \delta[\tau^2 - BF(x)] \\ &= -\frac{\omega_c \beta}{32\pi a^3} \int_0^\infty \mathrm{d}x\, x^3 \mathrm{e}^{-x} F^{-\frac{1}{2}}(x) (\mathrm{e}^{\frac{1}{2}\beta\omega_c\sqrt{F(x)}} - 1)^{-1}. \end{aligned} \tag{24}$$

The corresponding characteristic frequency $\omega_c = \omega_p\sqrt{d/a}$ of the Casimir interaction is less than the parameter $kT/\hbar$ ($\beta\omega_c \ll 1$) for sufficiently small thicknesses d. In this case

$$\Delta f(T) = -\zeta(3)\frac{kT}{8\pi a^3} - f_0 + \cdots \tag{25}$$

where the symbol $\cdots$ denotes terms of higher order of the smallness. Thus,

$$f(T) = f_0 + \Delta f(T) = -\zeta(3)\frac{kT}{8\pi a^3} + \cdots. \tag{26}$$

Asymptotics (26) can be shown to be valid at $B, C \ll 1$ as well as for $B, 1 \ll C$. In terms of the characteristic frequencies these inequalities give

$$\omega_c \ll \max\left(\frac{kT}{\hbar}, \sqrt{\nu\frac{kT}{\hbar}}\right). \tag{27}$$

If the inverse inequality is fulfilled we obtain the evaluation equation (7).

The surprising things are that formula (26) coincides with equation (5) for the Casimir force of the bulk metals and that $f(T)$ does not disappear even at $d \to 0$. The difference between the cases of the bulk materials and the thin films consists only in the inequality defining the high-temperature regime. This is the specific feature of the Casimir attraction of the *metals*. In contrast to dielectrics, a significant reduction of the characteristic frequencies of the thermal fluctuations in a metal film occurs if the film thickness decreases. In its turn, this leads to the growth of the effective dielectric permittivity (16) of the conductor and, as is evident from equation (18), to a disappearance of the dependence of the temperature Green function, regΓ, on the sample thickness. It is precisely this fact that results eventually in the unexpected insensitivity of the Casimir force to the thickness d in the high-temperature regime (27).

Using the asymptotics (7) and (26), we can obtain the following evaluating formula for the Casimir force:

$$f \propto -\left(kT + \frac{\hbar\omega_c^2}{\nu + \omega_c}\right)\frac{1}{a^3}. \tag{28}$$

It is necessary to keep in mind that equation (28) is obtained under condition (6) for the frequency ω_c and for sufficiently thin films with d satisfying the inequality (19).

4. Discussion

The theoretical description of the temperature dependence of the Casimir force between a bulk conductor and a thin metal film is given in this letter. In the general case, the Casimir force can be presented as a sum (12) of temperature-dependent and vacuum contributions of fluctuating electromagnetic field. We have obtained the surprising result for the situation of sufficiently thin films (or sufficiently high temperatures), equation (27). The Casimir force equation (26) has proved to be independent of the sample thickness in the main approximation. This fact is characteristic precisely for metals because the Casimir force for dielectric films with a constant value of ϵ vanishes at $d \to 0$.

Mathematically, the above-mentioned d independence of the Casimir attraction of the metal film in the regime (27) is connected to the proportionality of the characteristic frequency ω_c of the thermal fluctuations to the thickness of the film and to the significant reduction of ω_c with the decrease of d. The existence of the characteristic low-frequency regime in the Casimir attraction of sufficiently thin metal films is physically caused by the strong classical long-wavelength fluctuations of the conduction current and the plasmic shielding of the electromagnetic modes. These peculiarities add to the list of the characteristic features [20,21] of the Casimir effect for metals.

Let us discuss the possibility of observing the temperature contributions to the Casimir force. The high-temperature regime (27) can be realized for films of semimetals (such as Bi, Sb, As) with electron density $n \approx 10^{18}$ cm^{-3} and plasma frequency $\omega_p \approx 10^{14}$ s^{-1}. The Casimir force for films with $d \approx 10^{-6}$ cm, and $\nu \approx 10^{12}$ s^{-1} at separation of about 0.1 μm and $T = 300$ K is described by the asymptotics (26) and measures approximately 2 dyn cm^{-2}. For metals with the electron density $n \approx 10^{23}$ cm^{-3} and the plasma frequency $\omega_p \approx 10^{16}$ s^{-1}, the high-temperature regime (27) is practically unattainable. Therefore, only small temperature-dependent corrections to the Casimir force can be observed. For the low-temperature regime $\omega_c \gg kT/\hbar$, the main term in the Casimir force is described by equation (7). The temperature correction may be obtained from equation (21). Under conditions of the 'infra-red skin effect', $\nu \ll \omega_c$, the following asymptotics is valid,

$$f(T) = -0.0175\hbar\omega_p \frac{\sqrt{ad}}{a^4} - \frac{\pi^2\hbar\nu}{6da^2}\left(\frac{kT}{\hbar\omega_p}\right)^2. \tag{29}$$

The Casimir force for copper films with $d \approx 10^{-6}$ cm, $\nu \approx 10^{14}$ s^{-1} at separation of about 0.1 μm and $T = 300$ K measures approximately 10 dyn cm^2. The 30% relative increase of the temperature results in about 3% relative change in the Casimir force.

In conclusion, let us emphasize the entropy origin of the Casimir force (26). As follows from (26), the free energy of the Casimir interaction is of the form

$$\mathcal{F} = -\zeta(3)\frac{kT}{16\pi a^2}A. \tag{30}$$

Hence, the entropy of interaction is

$$S = 1.2\frac{A}{16\pi a^2}. \tag{31}$$

In contrast to the free energy and entropy, the energy E of interaction calculated by the formula

$$E = \frac{\partial}{\partial\beta}(\beta\mathcal{F}) \tag{32}$$

vanishes at $d \to 0$.

Appendix. Summation formula for the Casimir force

We transform the sum equation (10) over Matsubara's frequencies ω_n into the 'spectral' integrals using the Abel–Plana formula for summing a series:

$$\lim_{n\to\infty}\left\{\sum_{k=1}^{n} F(k) - \int_\theta^{n+\theta} \mathrm{d}x\, F(x)\right\}$$
$$= \int_\theta^{\theta-\mathrm{i}\infty} \mathrm{d}z\, F(z)(\mathrm{e}^{2\mathrm{i}\pi z}-1)^{-1} + \int_\theta^{\theta+\mathrm{i}\infty} \mathrm{d}z\, F(z)(\mathrm{e}^{-2\mathrm{i}\pi z}-1)^{-1} \tag{A.1}$$

where the function $F(z)$ is regular in the half-plane $\mathrm{Re}\, z > 0$ and satisfies the inequality

$$|F(x+\mathrm{i}y)| < f(x)\mathrm{e}^{a|y|} \qquad (a < 2\pi).$$

The function $f(x)$ is bounded at $x \to \infty$; $0 < \theta < 1$. The applicability of the Abel–Plana formula (A.1) to the expression for Casimir's force (10) is ensured by the analytical properties of the conductivity $\sigma^{(M)}$ and of the Green function $\Gamma^{(M)}$ in the upper half-plane of the complex frequency [22, 23]. Then, we make the limiting transition $\theta \to 0^+$ in equation (A.1) (0^+ is an infinitesimal positive parameter). The final transformations are carried out with regard to the formulae

$$\Gamma^{(M)}(\omega_n) = \Gamma(\mathrm{i}|\omega_n|) \qquad \Gamma(\omega) = \Gamma^{(M)}\left(\frac{\omega}{\mathrm{i}} + 0^+\right) \tag{A.2}$$

where Γ is the retarding Green function of the electromagnetic field, and the symmetry relations

$$\sigma^*(-\omega) = \sigma(\omega) \qquad \Gamma^*(-\omega) = \Gamma(\omega). \tag{A.3}$$

As a result, we obtain equation (12) for the Casimir force expressed in terms of the spectral integrals.

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